

The generation of spiral density waves by MRI in accretion discs

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ABSTRACT

We investigate the linear dynamics of non-axisymmetric perturbations in Keplerian discs subject to a weak uniform vertical magnetic field in the shearing box approximation. Perturbations are decomposed into shearing waves and evolved by numerically integrating the linearized ideal MHD equations. The disc flow supports three basic perturbation modes: two incompressible modes — magnetic mode that undergoes magnetorotational instability (MRI) and inertia–magnetic waves — and compressible spiral density waves. The magnetic mode and inertia–magnetic waves have a low frequency of the order of Alfvén and orbital frequencies, respectively, while density waves have high-frequency. We introduce mode eigenfunctions and governing modal equations to analyze the dynamics of individual modes. For non-axisymmetric modes, the modal equations are coupled due to the shear of the Keplerian rotation of the disc, giving rise to a new shear-induced linear mode coupling process, which is rooted in the non-self-adjoint nature of shear flows. We focus on the generation of density waves by the dominant MRI-unstable magnetic mode. We show that initially imposed magnetic mode undergoes MRI growth and abruptly excites density waves when its radial wavenumber crosses zero. The density wave–MRI coupling is most efficient when the azimuthal and vertical wavelengths of perturbations are comparable to the disc scale height. Since density waves are compressible, whereas MRI is incompressible, this wave excitation process can also be regarded as a linear mechanism generating compressible motions via MRI-driven incompressible ones. Its implications for compressible nonzero net vertical field MRI-turbulence are also discussed.

Key words: accretion, accretion discs – magnetohydrodynamics (MHD) – instabilities – (stars:)planetary systems: protoplanetary discs – turbulence

1 INTRODUCTION

Understanding angular momentum and mass transport in accretion discs is one of the central problems in astrophysics, which has been substantially advanced over the past decade. Different nonmagnetic thermal processes, such as subcritical baroclinic instability (e.g., Klahr & Bodenheimer 2003; Lesur & Papaloizou 2010; Lyra & Klahr 2011; Klahr et al. 2026) and related convective overstability (e.g., Klahr & Hubbard 2014; Lyra 2014; Latter 2016; Teed & Latter 2021, 2025), Rossby wave instability (e.g., Lovelace et al. 1999; Meheut et al. 2012; Lyra & Mac Low 2012; Miranda et al. 2016; Ono et al. 2018), vertical shear instability (e.g., Nelson et al. 2013; Stoll & Kley 2014; Barker & Latter 2015; Manger et al. 2020; Shariff & Umurhan 2024; Lesur et al. 2025; Ogilvie et al. 2025) as well as magnetohydrodynamic (MHD) processes, which are first of all magnetorotational instability (MRI, Balbus & Hawley 1991, 1992), Hall-shear instability (e.g., Kunz 2008; Kunz & Lesur 2013; Lesur et al. 2014; Simon et al. 2015; Béthune et al. 2017) and disc winds (e.g., Wang et al. 2019; Gressel et al. 2020; Lesur 2021) been explored in greater detail and characterized their ability to transport angular momentum (see a recent review Lesur et al. 2023).

Nevertheless, the MRI remains one of the most robust mechanisms for destabilizing weakly magnetized Keplerian discs and driving MHD turbulence that provides angular momentum transport at rates broadly consistent with observationally inferred accretion timescales (e.g. Meheut et al. 2015; Salvesen et al. 2016; Flock et al. 2017; Ross & Latter 2018; Gogichaishvili et al. 2018; Begelman & Armitage 2023).

In addition to turbulence, spiral density waves (SDWs) play a significant role in driving dynamical activity and angular momentum transport in discs as well as determining their morphology (Muto et al. 2012; Benisty et al. 2015; Pérez et al. 2016; Dong et al. 2018; Huang et al. 2019). They arise from the gas compressibility and can be excited by various mechanisms, including self-gravity (e.g., Goldreich & Tremaine 1978; Durisen et al. 2007; Kratter & Lodato 2016; Béthune et al. 2021, and references therein), vortices (Bodo et al. 2005; Mamatsashvili & Chagelishvili 2007; Mamatsashvili & Rice 2009; Heinemann & Papaloizou 2009a; Paardekooper et al. 2010) and rotating planets embedded in the disc (Goodman & Rafikov 2001; Rafikov 2002; Bae & Zhu 2018; Cimerman & Rafikov 2021). SDWs are capable of enhancing outward transport of angular momentum caused by other mechanisms and are even a dominant mode responsible for this transport in self-gravitating discs (Kratter & Lodato 2016). Moreover, SDWs generated in the active layers

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in protoplanetary discs can penetrate into the poorly ionized “dead zones”, where they can in turn drive angular momentum transport (Oishi & Mac Low 2009). In disc–planet interactions, a planet excites a pair of SDWs that propagate radially inward and outward in the disc. These waves exert torques on the planet, modifying its migration rate.

The dynamics of SDWs in magnetized discs and, in particular, their interplay with MRI has also been considered. Heinemann & Papaloizou (2009b) studied the excitation of SDWs in MRI-turbulence in discs with zero net magnetic flux in the shearing box approximation. In this case, the SDW generation is in fact hydrodynamic rather than MHD by nature, has a linear origin and is associated with a vertically uniform vortical mode with non-zero potential vorticity that acts as a source of SDWs, consistent with their original linear analysis (Heinemann & Papaloizou 2009a, hereafter HP09a). Using a temporal WKB approach and matching the asymptotic solutions for the initially imposed vortical mode to that for the subsequently excited SDWs, these authors quantified the efficiency of SDW generation by the vortical mode as a function of azimuthal wavenumber. It turned out to be most efficient when the azimuthal wavenumber of SDWs is comparable to the disc scale height, while these waves, like the vortical mode, do not vary vertically. On the other hand, magnetic field perturbations appeared to play only a minor role in the wave generation process. However, this might be due to the specific configuration considered in HP09a – a weak magnetic field with zero net flux – which prevents the development of more dominant, large-scale, exponentially growing MRI modes.

Large-scale regular wave structures nearly uniform along the vertical direction have been often observed in the density field of compressible MRI-driven turbulence with nonzero net flux in other simulations (e.g., Fromang & Stone 2009; Blaes et al. 2011; Gressel et al. 2012; Bai & Stone 2013; Zhu et al. 2013; Ross & Latter 2018; Sun & Bai 2021). These wave-like density features have been attributed to SDWs produced by vortex-wave coupling mechanism of HP09a in MRI-turbulence. However, unlike in the zero net flux case, growing MRI modes in the nonzero net flux case can dominate the vortical mode, thereby influencing SDW generation and dynamics. The detailed mechanism underlying the coupling between magnetic field perturbations and SDWs is still not well understood.

The linear coupling of vortices and SDWs in discs revealed by HP09a is in fact a special manifestation of a general linear mode coupling phenomenon in shear flows that arises from their non-self-adjoint nature (Trefethen et al. 1993). The non-self-adjointness of shear flows and its first consequence – transient, or nonmodal growth of perturbations were well understood and described in detail by the hydrodynamic community in the 1990s (Schmid & Henningson 2001; Schmid 2007, and references therein). Later, the relevance of the nonmodal growth phenomenon in the dynamics of accretion discs, which are a special type of shear flows, and transition to turbulence therein was also demonstrated (Lominadze et al. 1988; Chagelishvili et al. 2003; Zhuravlev & Razdoburdin 2014). A second significant consequence of non-self-adjointness of shear flows is the linear mode coupling phenomenon, which was revealed first as a coupling of vortical and sound waves in hydrodynamic shear flows (Chagelishvili et al. 1997) and later as a coupling between different wave branches in MHD shear flows (Chagelishvili et al. 1996; Gogoberidze et al. 2004). It plays a central role in the subsequent nonlinear evolution of perturbations and largely defines the characteristics of the resulting nonlinear state (turbulence).

The shear-induced linear mode coupling phenomenon inevitably

occurs in discs because of their strong Keplerian differential rotation, or shear. Indeed, several manifestations of this coupling process in discs have been analyzed in detail. First of all it is the generation of SDWs by vortices in Keplerian discs, which was originally studied by Bodo et al. (2005) both in local and global 2D non-self-gravitating discs and generalized by Mamatsashvili & Chagelishvili (2007) to self-gravitating discs, where it is much more efficient. Later, as mentioned above, this process was discussed in zero net flux MRI-turbulence in magnetized discs by Heinemann & Papaloizou (2009b). Other notable manifestations of the linear mode coupling in discs include generation of inertia-gravity waves by vortices (Tevzadze et al. 2003, 2008), generation of SDWs by entropy and vortical modes in the presence of baroclinic instability (Tevzadze et al. 2010), generation of SDWs by convection (Mamatsashvili & Rice 2011) and generation of inertia-gravity waves by MRI-unstable modes (Mamatsashvili et al. 2013).

In this paper, we study the linear dynamics and coupling of SDW and MRI modes in compressible Keplerian discs subject to a vertical uniform background magnetic field in the shearing box approximation. This extends, on the one hand, the work of HP09a to the case of nonzero net vertical magnetic flux and, on the other hand, our previous study Mamatsashvili et al. (2013), in which we demonstrated the linear coupling between inertial–gravity waves and MRI modes in incompressible stratified discs. The zero and nonzero net flux configurations are two main ones where MRI is usually studied. The MRI dynamics is different in these two cases. In the first case, there is a well-defined linear instability leading to the exponential growth of axisymmetric (channel) modes, which are most unstable, and the transient, or nonmodal growth of non-axisymmetric modes over a finite time (Goodman & Xu 1994; Pessah & Goodman 2009; Latter et al. 2009, 2010; Mamatsashvili et al. 2013). On the other hand, in the second case, MRI sets in as a subcritical instability and nonlinear dynamo action must be effective to sustain the large-scale field that in turn supports MRI (Herault et al. 2011; Riols et al. 2017; Rincon 2019). As a result, MRI is stronger in the nonzero net flux case – it is characterized by higher growth rate and hence nonlinearly saturates into MHD turbulence with a higher transport rate – than that in the zero-net flux case. This should clearly have an effect on the SDW generation process, since the main source responsible for wave generation can be magnetic field rather than potential vorticity, which is anyway not conserved in the 3D MHD case. By contrast, as shown by Heinemann & Papaloizou (2009b), in zero net flux MRI-turbulence, a magnetic field was not actually important in the SDW generation process, which, as mentioned above, is essentially of hydrodynamic nature stemming from potential vorticity. We show here that non-axisymmetric modes, undergoing nonmodal MRI growth, can excite high-frequency SDWs due to shear-induced linear mode coupling process and identified a range of azimuthal and vertical wavenumbers where this SDW-MRI coupling is most efficient.

The paper is organized as follows. In Section 2, we describe the mathematical formalism – linearize basic ideal MHD equations, classify modes and cast those equations into a system of six first order differential equations for mode eigenfunctions. Section 3 is devoted to the dynamics of non-axisymmetric modes, demonstrating the generation of SDWs by the magnetic mode and examining its efficiency as a function of wavenumbers. Summary and discussion are given in Section 4.

2 PHYSICAL MODEL AND MAIN EQUATIONS

We use the shearing box approximation (Goldreich & Lynden-Bell 1965; Hawley et al. 1995) to study the dynamics of perturbations in a magnetized Keplerian disc. In this model, gas motion is considered in a local Cartesian frame (x, y, z) , where x is the radial, y is the azimuthal and z is the vertical coordinate with the corresponding unit vectors $(\hat{x}, \hat{y}, \hat{z})$ in these directions. This frame co-rotates with the disc's angular velocity Ω at some fiducial radius r_0 from the central object. In this local approach, curvature effects due to cylindrical geometry of the disc are ignored. For simplicity, we neglect stratification due to the vertical gravity of the central object, because it is not important for the shear-induced mode coupling phenomenon studied here. In this unstratified shearing box, the equations of compressible ideal MHD are (e.g., Hawley et al. 1995):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2\Omega \hat{z} \times \mathbf{u} = -\frac{1}{\rho} \nabla \left(p + \frac{\mathbf{B}^2}{8\pi} \right) + \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi\rho} + 2q\Omega^2 x \hat{x} \quad (2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

where $\mathbf{u}$ is the velocity in the local frame, ρ and p are, respectively, the gas density and pressure, and $\mathbf{B}$ is the magnetic field. We close this system with an isothermal equation of state for the gas, $p = c_s^2 \rho$, where c_s is the constant sound speed.

In this rotating frame, the unperturbed Keplerian rotation of the disc represents an azimuthal flow $\mathbf{u}_0 = -q\Omega x \hat{y}$ with a constant velocity shear in the radial direction, where the shear parameter $q = 1.5$ for the Keplerian rotation. The equilibrium density ρ_0 and the pressure p_0 are spatially uniform due to the absence of vertical gravity. The flow is subject to a uniform vertical background magnetic field $\mathbf{B}_0 = B_{0z} \hat{z}$, which is assumed to be weak (sub-thermal) in the sense that the corresponding Alfvén speed, $v_A = B_{0z}/(4\pi\rho_0)^{1/2}$, is much smaller than the sound speed c_s , i.e., the usual plasma $\beta = 2c_s^2/v_A^2 \gg 1$. Below we take $\beta = 400$ comparable to that adopted in MRI studies in discs.

2.1 Perturbation equations

Consider small perturbations of density $\rho' = \rho - \rho_0$, velocity $\mathbf{u}' = \mathbf{u} - \mathbf{u}_0$ and magnetic field $\mathbf{b}' = \mathbf{B} - \mathbf{B}_0$ about the equilibrium state described above. Linearizing equations (1)-(4), we obtain the following system that governs the perturbation dynamics

$$\frac{D\rho'}{Dt} + \rho_0 \nabla \cdot \mathbf{u}' = 0, \quad (5)$$

$$\frac{Du'_x}{Dt} = -\frac{1}{\rho_0} \frac{\partial}{\partial x} \left(c_s^2 \rho' + \frac{B_{0z} b'_z}{4\pi} \right) + 2\Omega u'_y + \frac{B_{0z}}{4\pi\rho_0} \frac{\partial b'_x}{\partial z}, \quad (6)$$

$$\frac{Du'_y}{Dt} = -\frac{1}{\rho_0} \frac{\partial}{\partial y} \left(c_s^2 \rho' + \frac{B_{0z} b'_z}{4\pi} \right) + (q-2)\Omega u'_x + \frac{B_{0z}}{4\pi\rho_0} \frac{\partial b'_y}{\partial z}, \quad (7)$$

$$\frac{Du'_z}{Dt} = -\frac{1}{\rho_0} \frac{\partial}{\partial z} \left(c_s^2 \rho' + \frac{B_{0z} b'_z}{4\pi} \right) + \frac{B_{0z}}{4\pi\rho_0} \frac{\partial b'_z}{\partial z}, \quad (8)$$

$$\frac{Db'_x}{Dt} = B_{0z} \frac{\partial u'_x}{\partial z}, \quad (9)$$

$$\frac{Db'_y}{Dt} = B_{0z} \frac{\partial u'_y}{\partial z} - q\Omega b'_x, \quad (10)$$

$$\frac{\partial b'_x}{\partial x} + \frac{\partial b'_y}{\partial y} + \frac{\partial b'_z}{\partial z} = 0, \quad (11)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} - q\Omega x \frac{\partial}{\partial y},$$

is the derivative along the base Keplerian shear flow. The form of equations (5)-(11) permits a decomposition of the perturbations into shearing plane waves with time-dependent amplitudes and phases, which are the natural basis functions in the shearing box (Goldreich & Lynden-Bell 1965; Hawley et al. 1995)

$$F(\mathbf{r}, t) = \bar{F}(t) \exp[ik_x(t)x + ik_y y + ik_z z], \quad (12)$$

where $F \equiv (\rho', \mathbf{u}', \mathbf{b}')$ denotes the perturbations and $\bar{F} \equiv (\bar{\rho}, \bar{\mathbf{u}}, \bar{\mathbf{b}})$ their Fourier amplitudes. The azimuthal, k_y , and vertical, k_z , wavenumbers do not change with time, while the radial wavenumber $k_x(t) = q\Omega k_y t$ varies with time due to shear at a constant rate $q\Omega k_y$ for non-axisymmetric modes with $k_y \neq 0$. For convenience, in $k_x(t)$ we have shifted the origin of time towards negative values, so that $k_x(0) = 0$ at $t = 0$. In this case, any initially leading shearing wave with $k_x(t)/k_y < 0$ at $t < 0$, eventually changes (swings) to a trailing orientation with $k_x(t)/k_y > 0$ at $t > 0$ as its k_x increases.

Substituting (12) into equations (5)-(11), we arrive at the following system of first order ordinary differential equations that govern the linear dynamics of Fourier amplitudes of perturbations

$$\frac{d\bar{\rho}}{dt} + i\rho_0 [k_x(t)\bar{u}_x + k_y\bar{u}_y + k_z\bar{u}_z] = 0, \quad (13)$$

$$\frac{d\bar{u}_x}{dt} = -\frac{ik_x(t)}{\rho_0} \left(c_s^2 \bar{\rho} + \frac{B_{0z} \bar{b}_z}{4\pi} \right) + 2\Omega \bar{u}_y + \frac{ik_z B_{0z}}{4\pi\rho_0} \bar{b}_x, \quad (14)$$

$$\frac{d\bar{u}_y}{dt} = -\frac{ik_y}{\rho_0} \left(c_s^2 \bar{\rho} + \frac{B_{0z} \bar{b}_z}{4\pi} \right) + (q-2)\Omega \bar{u}_x + \frac{ik_z B_{0z}}{4\pi\rho_0} \bar{b}_y, \quad (15)$$

$$\frac{d\bar{u}_z}{dt} = -\frac{ik_z}{\rho_0} c_s^2 \bar{\rho}, \quad (16)$$

$$\frac{d\bar{b}_x}{dt} = ik_z B_{0z} \bar{u}_x, \quad (17)$$

$$\frac{d\bar{b}_y}{dt} = ik_z B_{0z} \bar{u}_y - q\Omega \bar{b}_x, \quad (18)$$

$$k_x(t)\bar{b}_x + k_y\bar{b}_y + k_z\bar{b}_z = 0. \quad (19)$$

Eliminating velocities from equations (16)-(18), we rewrite

equations (13)–(19) only for the density $\bar{\rho}$ and the field components $\bar{b}_x$ and $\bar{b}_y$ (henceforth bars over the Fourier amplitudes will be omitted),

$$\frac{d^2\rho}{dt^2} + c_s^2 k^2(t)\rho = (v_A^2 k^2(t) - 2q\Omega^2) \frac{k_x(t)}{k_z} b_x + v_A^2 k^2(t) \frac{k_y}{k_z} b_y - 2\Omega \frac{k_x(t)}{k_z} \frac{db_y}{dt} - 2(q-1)\Omega \frac{k_y}{k_z} \frac{db_x}{dt}, \quad (20)$$

$$\frac{d^2 b_x}{dt^2} = 2\Omega \frac{db_y}{dt} + [2q\Omega^2 - v_A^2(k_x^2(t) + k_z^2)] b_x - v_A^2 k_x(t) k_y b_y + c_s^2 k_x(t) k_z \rho, \quad (21)$$

$$\frac{d^2 b_y}{dt^2} = -2\Omega \frac{db_x}{dt} - v_A^2(k_y^2 + k_z^2) b_y - v_A^2 k_x(t) k_y b_x + c_s^2 k_y k_z \rho, \quad (22)$$

where $k^2(t) = k_x^2(t) + k_y^2 + k_z^2$ and without loss of generality we assume $k_y, k_z \geq 0$. Equations (20)–(22) form the basis for our subsequent analysis. They describe the dynamics and coupling of different perturbation modes existing in the considered compressible Keplerian flow with a vertical background field in the shearing box model. We will classify these modes below using the Wentzel–Kramers–Brillouin (WKB) approach with respect to time.

Combining equations (19), (21) and (22), we can further simplify equation (20) to a more compact form,

$$\frac{d^2\rho}{dt^2} + c_s^2 k_z^2 \rho = \frac{d^2 b_z}{dt^2}, \quad (23)$$

which will be useful in the following WKB analysis.

2.2 WKB approach and classification of modes

The shear q enters equations (20)–(22) implicitly through the time-dependent radial wavenumber $k_x(t) = q\Omega k_y t$ only for non-axisymmetric, $k_y \neq 0$, modes and explicitly on the right hand side of equations (20) and (21). Therefore, we can distinguish between two types of shear-induced effects. The implicit dependence on the shear leads to finite-time linear phenomena – transient, or nonmodal growth and linear mode coupling – in the dynamics of non-axisymmetric perturbations that arise from the drift of the time-varying $k_x(t)$ along the k_x -axis and vanish for axisymmetric, $k_y = 0$, perturbations. The explicit dependence on the shear, on the other hand, affects both axisymmetric and non-axisymmetric perturbations. It sets the frequency of epicyclic oscillations and, importantly, is responsible for the existence of MRI in differentially rotating magnetized discs (Balbus & Hawley 1991, 1998).

To classify the modes present in the given magnetized disc flow, we first consider the case of large wavenumbers $kH \gg 1$, where $H = c_s/\Omega$ is the disc scale height, and $k_y \ll k_z$ when the WKB, or adiabatic approximation holds uniformly at all times (Balbus & Hawley 1992; Johnson 2007). This means that the time-variation of k_x and hence the implicit dependence on the shear in equations (20)–(22) can be neglected. We will see below that at larger k_y and smaller $k_x(t)$ which ensues eventually as $k_x(t)$ decreases in absolute value and crosses the point $k_x = 0$, the mode dynamics is no longer adiabatic and hence nonmodal shear-induced effects come into play.

In this WKB regime, we look for solutions with a standard modal form $\rho, b_x, b_y \propto \exp(-i \int^t \omega(t') dt')$. Substituting it into equations (21)–(23), using equations (19) to eliminate b_z and assuming that the time-dependent frequency, $\omega(t)$, satisfies the

adiabatic approximation $|d\omega(t)/dt| \ll \omega^2(t)$, we obtain the linear algebraic equations

$$\omega^2(k_x b_x + k_y b_y) + k_z(\omega^2 - c_s^2 k_z^2)\rho = 0, \quad (24)$$

$$[\omega^2 + 2q\Omega^2 - v_A^2(k_x^2 + k_z^2)] b_x - (2i\omega\Omega + v_A^2 k_x k_y) b_y + c_s^2 k_x k_z \rho = 0, \quad (25)$$

$$(2i\omega\Omega - v_A^2 k_x k_y) b_x + [\omega^2 - v_A^2(k_y^2 + k_z^2)] b_y + c_s^2 k_y k_z \rho = 0. \quad (26)$$

Equating the determinant of this linear system to zero yields the following sixth-order dispersion relation

$$\omega^6 - a_1 \omega^4 + a_2 \omega^2 + a_3 = 0, \quad (27)$$

with the coefficients

$$a_1 = 2(2 - q)\Omega^2 + (c_s^2 + v_A^2) k^2 + v_A^2 k_z^2,$$

$$a_2 = v_A^2(v_A^2 + 2c_s^2)k_z^2 k^2 + 4\Omega^2 c_s^2 k_z^2 - 2q\Omega^2(c_s^2 + v_A^2)(k_y^2 + k_z^2)$$

$$a_3 = c_s^2 v_A^2 k_z^2 [2q\Omega^2(k_y^2 + k_z^2) - v_A^2 k_z^2 k^2].$$

For horizontally uniform $k_x = k_y = 0$ perturbations, dispersion relation (27) coincides with that of the compressible single-fluid one in Blaes & Balbus (1994, see their equation 61). In the incompressible limit $c_s \rightarrow \infty$, at $k_y = 0$ it reduces to the well-known dispersion relation of Balbus & Hawley (1991) that describes the axisymmetric MRI (see also Johnson 2007; Mamatsashvili et al. 2013). When shear is absent $q = 0$, this dispersion relation describes classical MHD waves in a rotating fluid (Lehnert 1954). So, we can view equation (27) as a generalized WKB dispersion relation. It is a cubic polynomial for ω^2 with three distinct solutions, which we denote in ascending order by $\omega_m^2 < \omega_{im}^2 < \omega_s^2$, corresponding to three physically different types of perturbation modes. To clarify the nature of these modes, using the limit $kH \gg 1$, we derive approximate analytical solutions to equation (27). These solutions have simpler, more physically transparent forms and describe the following modes:

1. High-frequency compressive SDWs with

$$\omega_s^2 \approx a_1 = 2(2 - q)\Omega^2 + (c_s^2 + v_A^2) k^2 + v_A^2 k_z^2, \quad (28)$$

the restoring force for which is provided by pressure (compressibility), but is modified by rotation and magnetic field. Since the background field is weak ($c_s \gg v_A$), expression (28) reduces to the well-known local dispersion relation of SDWs in non-magnetized discs, $\omega_s^2 = 2(2 - q)\Omega^2 + c_s^2 k^2$ (Goldreich & Tremaine 1978; Balbus 2003).

2. Low-frequency inertia-magnetic waves (IMWs) with

$$\omega_{im}^2 = \frac{a_2 + \sqrt{a_2^2 + 4a_1 a_3}}{2a_1}, \quad (29)$$

the restoring force for which is provided by Coriolis and magnetic tension forces but is modified by compressibility. In the incompressible limit, for a weak magnetic field, $v_A k_z \ll \Omega$, and for $k_y = 0$, these waves become usual inertial waves with $\omega_{im}^2 = 2(2 - q)\Omega^2 k_z^2 / k^2$ (Balbus 2003; Johnson 2007), while for a strong magnetic field $v_A k_z \gg \Omega$ they oscillate with the Alfvén frequency, $\omega_{in} = v_A k_z$. Since $\omega_{im}^2 > 0$, IMWs are stable. At $k_z = 0$ they reduce to stationary, vertically uniform, hydrodynamic 2D vortical modes.

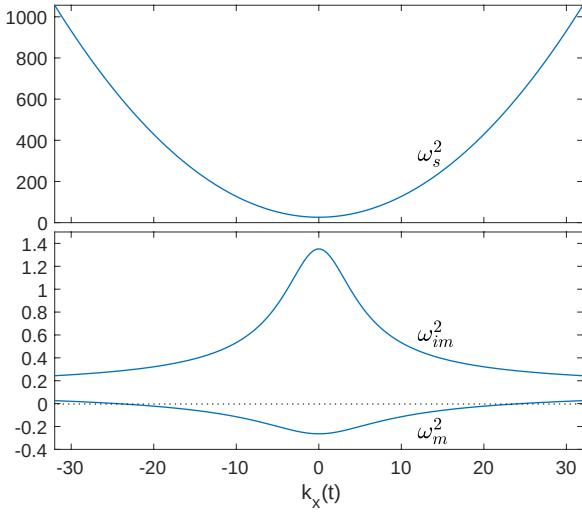


Figure 1. Squared frequencies ω_s^2 of SDWs, ω_{im}^2 of IMWs and ω_m^2 of the magnetic mode as a function of $k_x(t)$ at $k_y = 1, k_z = 5$. ω_m^2 is negative in the range of k_x given by equation (31), indicating MRI of the magnetic mode in this range with the growth rate $\gamma = \sqrt{-\omega_m^2}$, which reaches its maximum, $\gamma_m = 0.51$, at $k_x = 0$.

3. Magnetic mode with frequency/growth rate¹

$$\omega_m^2 = \frac{a_2 - \sqrt{a_2^2 + 4a_1a_3}}{2a_1}, \quad (30)$$

the restoring force for which is provided by a magnetic tension force but is modified by compressibility and rotation. That is, this mode arises from the presence of the background field, because, as follows from equation (30), $\omega_m^2 \propto v_A^2$ and therefore disappears when $v_A \rightarrow 0$. The magnetic mode plays an important dynamical role because it is the one that exhibits MRI. Indeed, it follows from equation (30) that the magnetic mode is unstable, i.e., $\omega_m^2 < 0$, if the coefficient $a_3 > 0$, which implies

$$v_A^2 k_z^2 < 2q\Omega^2, \quad k_x^2 < (2q\Omega^2 - v_A^2 k_z^2) \frac{k_y^2 + k_z^2}{v_A^2 k_z^2}. \quad (31)$$

On the other hand, at large $|k_x|$ and/or smaller shear q , when $a_3 < 0$, this mode is stable, $\omega_m^2 \geq 0$, reducing to Alfvén wave.

Figure 1 shows the squared frequencies of SDW, IMW and magnetic mode as a function of k_x (which itself changes with time) for a given k_y and k_z . ω_{im}^2 and ω_s^2 are always positive, i.e., these modes are stable, whereas $\omega_m^2 < 0$ in a certain range of k_x given by conditions (31), which indicates MRI of the magnetic mode with the growth rate $\gamma = \sqrt{-\omega_m^2}$, reaching its maximum at $k_x = 0$. At large $|k_x| \gg k_y, k_z$ the effects of rotation and shear become small in the dynamics of IMWs and the magnetic mode and their oscillation frequencies approach the Alfvénic frequency, $\omega_m \approx \omega_{im} = v_A k_z$.

In this figure and everywhere below, we non-dimensionalize time by Ω^{-1} , frequency by Ω , length by H , wavenumber by H^{-1} , velocity by c_s , density by ρ_0 and magnetic field by B_{0z} . The non-dimensional Alfvén speed is then $v_A = \sqrt{2/\beta}$.

Thus, perturbation modes in a compressible, unstratified, weakly magnetized disc can be classified into three basic types –

compressible SDWs and two incompressible IMW and the magnetic modes. Below we focus on the shear-induced linear coupling between the MRI-unstable, and hence the strongest, magnetic mode with SDWs. To this end, following Gogoberidze et al. (2004); Tevzadze et al. (2010); Mamatsashvili & Rice (2011); Mamatsashvili et al. (2013), we introduce modal eigenfunctions, which can be convenient when studying mode dynamics in the presence of shear.

Switching to new variables,

$$h_1 = \rho - b_z, \quad h_2 = \frac{d}{dt}(\rho - b_z), \quad h_3 = b_x,$$

$$h_4 = \frac{db_x}{dt}, \quad h_5 = b_y, \quad h_6 = \frac{db_y}{dt},$$

substituting them into equations (21)–(23) and using the WKB-type modal time-dependence given above, we obtain

$$\frac{d\mathbf{h}}{dt} = -i\omega\mathbf{h} = \mathbf{A} \cdot \mathbf{h}, \quad (32)$$

which is essentially the same as the linear system of algebraic equations (24)–(26), but rewritten in a matrix form with $\mathbf{h} = [h_1, h_2, h_3, h_4, h_5, h_6]^T$ being the state column vector and the evolution matrix $\mathbf{A}$ is given in Appendix A. The eigenvalues of the linear system (32) are found from

$$\text{Det}(\mathbf{A} + i\omega\mathbf{I}) = 0,$$

which yields the same dispersion relation (27) for ω and hence its roots, multiplied by $-i$, $\pm i\omega_m, \pm i\omega_{im}$ and $\pm i\omega_s$, represent the eigenvalues of the matrix $\mathbf{A}$. Since these eigenvalues are generally distinct and the corresponding eigenvectors are linearly independent, we can perform the eigendecomposition of $\mathbf{A}$ (Golub & Van Loan 1996),

$$\mathbf{A} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}^{-1}, \quad (33)$$

where $\mathbf{\Lambda}$ is the diagonal matrix whose elements are the corresponding eigenvalues

$$\mathbf{\Lambda} = \text{diag}(i\omega_m, -i\omega_m, i\omega_{im}, -i\omega_{im}, i\omega_s, -i\omega_s)$$

and the columns of the matrix $\mathbf{C}$ are the eigenvectors of $\mathbf{A}$ ordered such that the first column corresponds to the eigenvalues $i\omega_m$, second to $-i\omega_m$, third to $i\omega_{im}$, fourth to $-i\omega_{im}$, fifth to $i\omega_s$ and sixth to $-i\omega_s$, which are all explicitly given in Appendix B.

Substituting (33) into equation (32), we obtain a canonical form of this system that consists of a set of first order differential equations governing each of these three modes,

$$\frac{d\psi}{dt} = \mathbf{\Lambda} \cdot \psi, \quad (34)$$

where $\psi \equiv [\psi_m^{(+)}, \psi_m^{(-)}, \psi_{im}^{(+)}, \psi_{im}^{(-)}, \psi_s^{(+)}, \psi_s^{(-)}]^T = \mathbf{C}^{-1} \cdot \mathbf{h}$ are the eigenfunctions of the counter-propagating (denoted by ‘-’ and ‘+’) magnetic mode ($\psi_m^{(\pm)}$), IMWs ($\psi_{im}^{(\pm)}$) and SDWs ($\psi_s^{(\pm)}$), respectively. Therefore, the matrix $\mathbf{C}$ relating the fluid variables $\mathbf{h}$ to the eigenfunctions ψ is referred to as the transformation matrix. Because $\mathbf{\Lambda}$ is diagonal, these eigenfunctions are dynamically decoupled and evolve independently from each other with their corresponding eigenfrequencies,

$$\psi_m^{(\pm)} \propto e^{\pm i \int^t \omega_m(t') dt'},$$

$$\psi_{im}^{(\pm)} \propto e^{\pm i \int^t \omega_{im}(t') dt'},$$

$$\psi_s^{(\pm)} \propto e^{\pm i \int^t \omega_s(t') dt'}.$$

¹ In the literature, this mode is often called MRI mode, but we prefer to refer to it as a magnetic mode (see also Ogilvie 1998), because besides MRI at smaller k_x , it is stable at large $|k_x|$ and reduces to Alfvén wave.

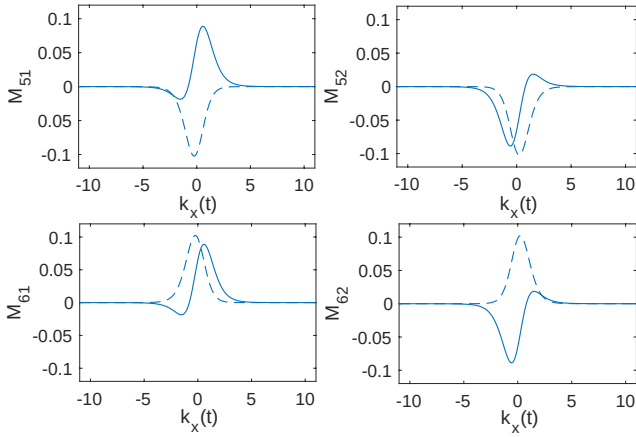


Figure 2. The non-diagonal elements M_{51}, M_{52} and M_{61}, M_{62} of the coupling matrix $\mathbf{M}$, which are responsible for the coupling of the magnetic mode and SDWs, as a function of $k_x(t)$ at $k_y = 2$, $k_z = 1$ (the solid lines show real and the dashed lines imaginary parts). These coupling coefficients vanish at large $|k_x(t)| \gg k_y$, in the WKB regime, where the dynamics is adiabatic and the modes are decoupled, but are appreciable in the non-adiabatic regime at $|k_x(t)| \lesssim k_y$, resulting in the mode interactions.

The absence of coupling among the modal equations (34) means that in the WKB regime, the modes evolve without exchanging energy among each other, i.e., initially exciting one mode with a specific time-scale of variation does not lead to the excitation of other modes with different time-scales. We will see below that in the non-WKB, or nonadiabatic regime, these modal equations become coupled due to shear, resulting in the dynamical coupling of the modes.

2.3 The modal equations in the non-WKB regime – the linear mode coupling

In the WKB regime considered above, the time variation of $k_x(t)$ and hence of the mode eigenfrequencies have been neglected. However, at higher $k_y \gtrsim k_z$, when $k_x(t)$ of non-axisymmetric modes, drifting along the k_x -axis, passes through the interval $|k_x| \lesssim k_y$, the WKB regime breaks down, $|d\omega(t)/dt| \sim \omega^2(t)$, and one should take into account the time-dependence of $k_x(t)$ in the main equations (20)–(22), or equivalently, in the $\mathbf{A}$ matrix in equation (32). This non-adiabatic regime is more important, since the shear-induced nonmodal effects manifest themselves in the mode dynamics in this regime. We now generalize the decoupled modal equations (34) to the non-WKB regime. In this case, equation (32) has the same form

$$\frac{d\mathbf{h}}{dt} = \mathbf{A}(t) \cdot \mathbf{h}, \quad (35)$$

but $\mathbf{A}$ now depends on time through $k_x(t) = q\Omega k_y t$ (see equation A1). Taking into account that the transformation matrix $\mathbf{C}$ also varies with time through $k_x(t)$, from equation (35) we obtain a more general system of linear modal equations both in the adiabatic and non-adiabatic regimes

$$\frac{d\psi}{dt} = [\mathbf{A}(t) + \mathbf{M}(t)] \cdot \psi, \quad (36)$$

where the generalized eigenfunctions are given, as before, by $\psi(t) = \mathbf{C}^{-1}(t) \cdot \mathbf{h}(t)$. The eigenvalue matrix $\mathbf{A}(t)$ has also become time-dependent through $k_x(t)$. The matrix

$$\mathbf{M}(t) = -\mathbf{C}^{-1}(t) \frac{d\mathbf{C}}{dt} \mathbf{C}(t) \quad (37)$$

is a new term compared to modal equations (34), which arises from the time-dependence of $k_x(t)$ and hence from shear q , because $\mathbf{M} \propto dk_x/dt = qk_y$, as follows from expression (37). In other words, this new matrix term is determined by shear q and is important for non-axisymmetric modes. To better see its role in the mode dynamics, it is convenient to write equation (36) componentwise

$$\begin{aligned} \frac{d\psi_m^{(+)}}{dt} - (i\omega_m + M_{11})\psi_m^{(+)} \\ = M_{12}\psi_m^{(-)} + M_{13}\psi_{im}^{(+)} + M_{14}\psi_{im}^{(-)} + M_{15}\psi_s^{(+)} + M_{16}\psi_s^{(-)}, \end{aligned} \quad (38)$$

$$\begin{aligned} \frac{d\psi_m^{(-)}}{dt} + (i\omega_m - M_{22})\psi_m^{(-)} \\ = M_{21}\psi_m^{(+)} + M_{23}\psi_{im}^{(+)} + M_{24}\psi_{im}^{(-)} + M_{25}\psi_s^{(+)} + M_{26}\psi_s^{(-)}, \end{aligned} \quad (39)$$

$$\begin{aligned} \frac{d\psi_{im}^{(+)}}{dt} - (i\omega_{im} + M_{33})\psi_{im}^{(+)} \\ = M_{31}\psi_m^{(+)} + M_{32}\psi_m^{(-)} + M_{34}\psi_{im}^{(-)} + M_{35}\psi_s^{(+)} + M_{36}\psi_s^{(-)}, \end{aligned} \quad (40)$$

$$\begin{aligned} \frac{d\psi_{im}^{(-)}}{dt} + (i\omega_{im} - M_{44})\psi_{im}^{(-)} \\ = M_{41}\psi_m^{(+)} + M_{42}\psi_m^{(-)} + M_{43}\psi_{im}^{(+)} + M_{45}\psi_s^{(+)} + M_{46}\psi_s^{(-)}, \end{aligned} \quad (41)$$

$$\begin{aligned} \frac{d\psi_s^{(+)}}{dt} - (i\omega_s + M_{55})\psi_s^{(+)} = \\ = M_{51}\psi_m^{(+)} + M_{52}\psi_m^{(-)} + M_{53}\psi_{im}^{(+)} + M_{54}\psi_{im}^{(-)} + M_{56}\psi_s^{(-)}, \end{aligned} \quad (42)$$

$$\begin{aligned} \frac{d\psi_s^{(-)}}{dt} + (i\omega_s - M_{66})\psi_s^{(-)} = \\ = M_{61}\psi_m^{(+)} + M_{62}\psi_m^{(-)} + M_{63}\psi_{im}^{(+)} + M_{64}\psi_{im}^{(-)} + M_{65}\psi_s^{(+)}. \end{aligned} \quad (43)$$

It is evident that the matrix $\mathbf{M}$ couples the mode eigenfunctions as well as modifies their frequencies, so we appropriately refer to it as a coupling matrix. All its elements, being proportional to q and k_y , are thus appreciable for non-axisymmetric modes and vanish in the WKB regime at large k and small $k_y \ll |k_x|, k_z$. The latter condition includes axisymmetric modes with $k_y = 0$, which are therefore not coupled in the linear regime. The coupling matrix $\mathbf{M}$ therefore encapsulates all shear-induced nonmodal effects. The left-hand sides of equations (38)–(43) describe the individual dynamics of the magnetic mode, IMWs and SDWs, respectively, modified by shear via the diagonal components of $\mathbf{M}$ that give rise to the shear-induced nonmodal growth of these modes. For the magnetic mode, MRI remains the dominant amplification mechanism, while nonmodal effects, mediated by M_{11} and M_{22} , further modify its dynamics. On the other hand, the right-hand sides, originating from the non-diagonal elements of $\mathbf{M}$, describe another main nonmodal effect due to shear – mutual couplings among these modes as well as the interaction between counter-propagating components of each mode. Specifically, the non-diagonal elements M_{12} and M_{21} , M_{34} and M_{43} , M_{56} and M_{65} describe the coupling, respectively, between the

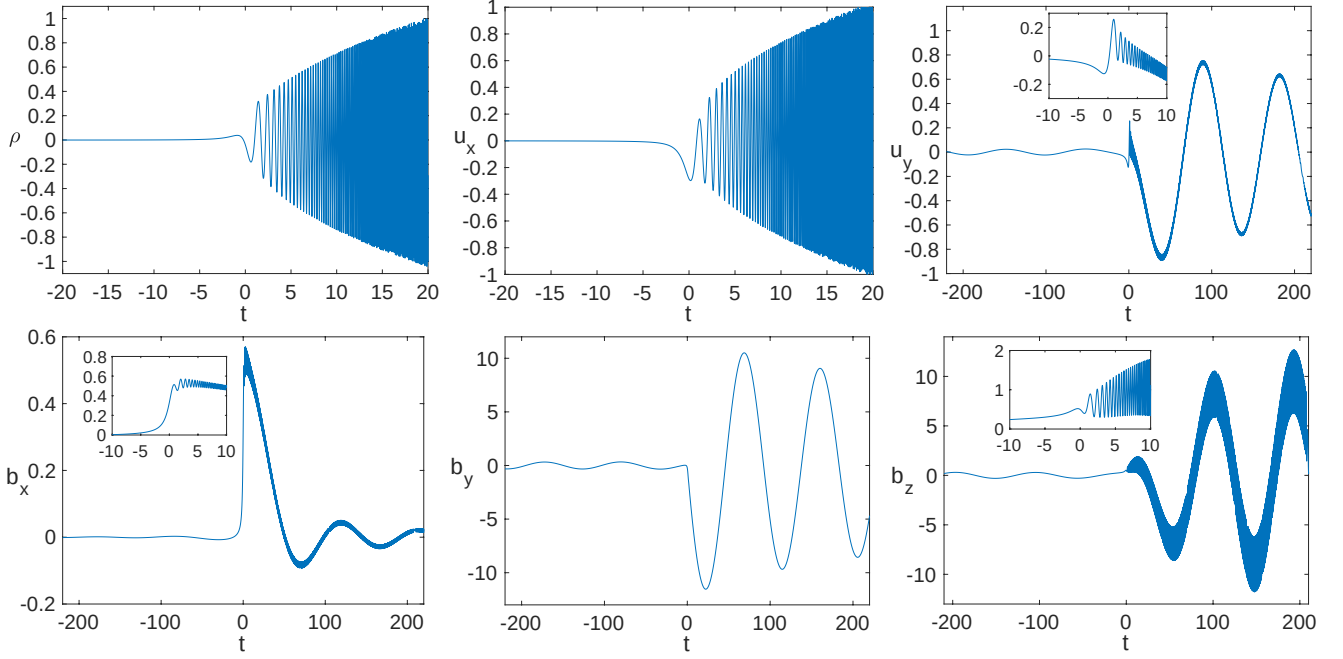


Figure 3. Evolution of the real parts of the density, velocity and magnetic field components pertaining to the initially imposed magnetic mode shearing wave with $k_y = 2$ and $k_z = 1$. The dominant components in the early evolution of the magnetic mode are transverse b_y and b_z , which oscillate with lower frequency ω_m close to the Alfvén frequency $v_A k_z$, while longitudinal components b_x are almost zero. Then, k_x enters the MRI-active range (31), where all three field components undergo nonmodal MRI growth. When k_x crosses the point $k_x = 0$ at $t = 0$, high-frequency oscillations abruptly emerge, which are strongest in ρ and u_x but also noticeable in b_x , u_y , b_y and b_z . After leaving the MRI-active interval, u_y , b_y and b_z , being mainly related to the original magnetic mode and generated IMWs, continue to oscillate with the lower-frequency and much higher amplitude at large times. The excited fast oscillations have high frequency ω_s and indicate the excitation of SDWs.

counter-propagating components, $\psi_m^{(+)}$ and $\psi_m^{(-)}$, of the magnetic mode, $\psi_{im}^{(+)}$ and $\psi_{im}^{(-)}$ of IMWs and $\psi_s^{(+)}$ and $\psi_s^{(-)}$ of SDWs. The other non-diagonal elements describe the couplings among different modes: $M_{13}, M_{14}, M_{23}, M_{24}$ and $M_{31}, M_{41}, M_{32}, M_{42}$ describe the coupling between the magnetic mode and IMWs, $M_{53}, M_{54}, M_{63}, M_{64}$ and $M_{35}, M_{45}, M_{36}, M_{56}$ describe the coupling between IMWs and SDWs and $M_{51}, M_{52}, M_{61}, M_{62}$ and $M_{15}, M_{25}, M_{16}, M_{26}$ describe the coupling between the SDWs and the magnetic mode. In other words, due to these coupling terms, an initially imposed one of these modes can act as a source for the other modes and excite them during evolution.

All these non-diagonal components reach their maximum and are appreciable in the non-adiabatic range $|k_x(t)| \lesssim |k_y|$, where the nonmodal effects are important (Tevzadze et al. 2010; Mamatsashvili & Rice 2011; Mamatsashvili et al. 2013). In our previous paper Mamatsashvili et al. (2013), we analyzed the nonmodal growth and coupling of inertia-gravity waves and magnetic mode in incompressible magnetized discs. Although the system of coupled equations (38)–(43) is quite complex involving all possible mode couplings, here we focus specifically on the coupling between SDWs and the dominant, MRI-unstable magnetic mode, which is determined by the coefficients $M_{51}, M_{52}, M_{61}, M_{62}$ shown in Fig. 2. They are appreciable in the non-adiabatic range $|k_x(t)| \lesssim k_y$, peaking around $k_x = 0$, and indicate that the shear-induced coupling between these modes occurs just in this range. At large $|k_x(t)| \gg k_y, k_z$, in the WKB regime, the coupling matrix vanishes, $\mathbf{M} \rightarrow 0$, and hence there is no energy exchange among the

modes.² Below, we characterize this SDW-MRI coupling in more detail.

3 EVOLUTION OF THE MODES – COUPLING OF MRI AND SDWS

In this section, we investigate the dynamics of the non-axisymmetric magnetic mode, its nonmodal MRI growth and generation of SDWs by numerically solving system (35). At large times $|t| \gg 1$, when $|k_x(t)| \gg k_y, k_z$, as discussed above, the WKB condition holds and the coupling matrix vanishes, implying that the modes are dynamically decoupled from each other. This allows us to impose only the magnetic mode in the beginning, as an initial condition for equations (35), and follow its dynamics.

To prepare such initial conditions, initially at $t = -t_0$, where $t_0 \gg 1$ is some large parameter, we impose a tightly leading ($k_x(-t_0)/k_y < 0, |k_x(-t_0)|/k_y \gg 1$) shearing wave of the purely magnetic mode and trace subsequent evolution until $t = t_0$, when shearing waves of the modes become tightly trailing ($k_x(t_0)/k_y \gg 1$). Without loss of generality, from the two counter-propagating components of the magnetic mode, we choose the one given by $\psi_m^{(-)}$ as an initial condition at $t = -t_0$. In this adiabatic regime, ignoring the mode coupling, this eigenfunction is given by the asymptotic WKB solution of the homogeneous, left hand side, part of equation

² In fact, other elements of the $\mathbf{M}$ matrix behave similarly with k_x , so we do not show them here.

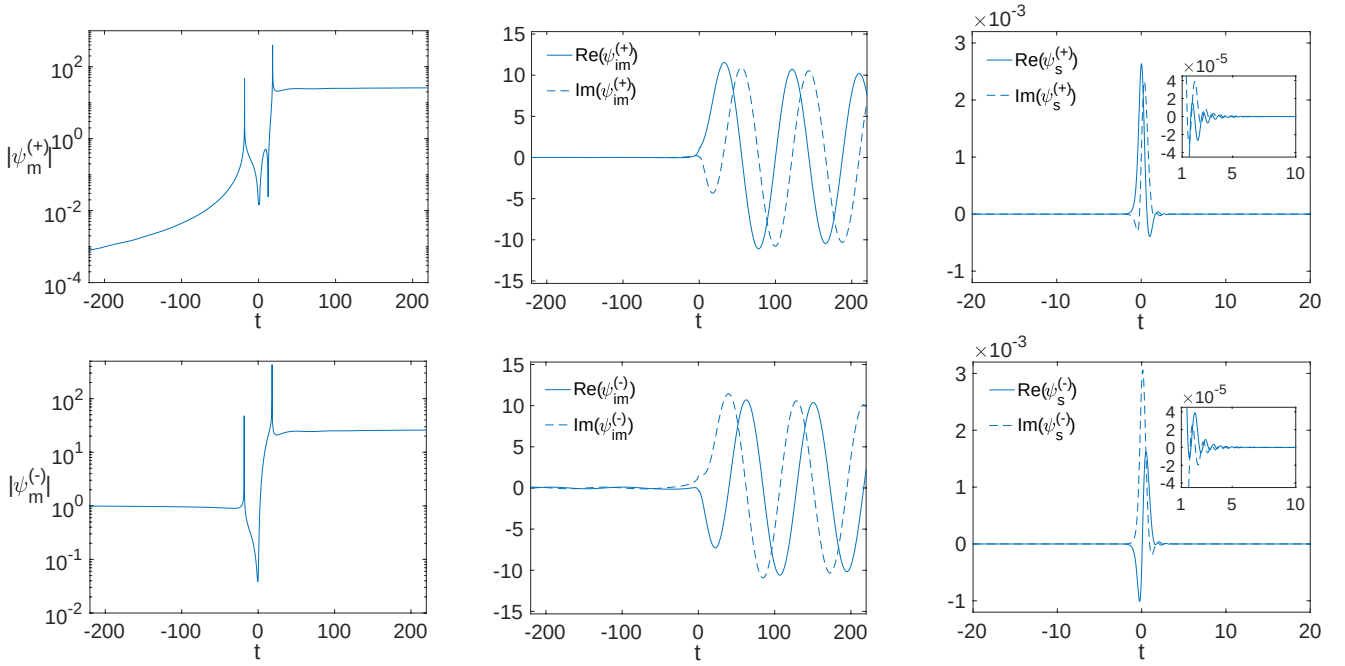


Figure 4. Evolution of the eigenfunctions at $k_y = 2$ and $k_z = 1$, which correspond to the perturbations in Fig. 3, for the initially imposed magnetic mode $\psi_m^{(-)}$. $\psi_m^{(-)}$, undergoing nonmodal MRI growth in the instability region $|t| \leq 18.3$ (see text), induces the other component $\psi_m^{(+)}$ via the coupling term M_{12} . The latter component in turn, feeding back on $\psi_m^{(-)}$ via the coupling term M_{21} , causes its decrease around $t = 0$ over a short period of time. Then, at $t > 0$ both $\psi_m^{(\pm)}$ increase (to highlight their growth, only the absolute values of $\psi_m^{(\pm)}$ are shown). On the other hand, $\psi_{im}^{(\pm)}$ and $\psi_s^{(\pm)}$ remain nearly zero at $t < 0$ and at about $t = 0$ display emergence of low-frequency (ω_{im}) and high-frequency (ω_s) oscillations, indicating the excitation of IMWs and SDWs, respectively.

(39),

$$\psi_m^{(-)} = C_0 e^{-\int_{-t_0}^t [i\omega_m(t') - M_{22}(t')] dt'} = e^{-i\nu_A k_z t} \quad (44)$$

where C_0 is some arbitrary constant setting the initial value, which is chosen such that at $t = -t_0$, $\psi_m^{(-)}$ has a unit amplitude and oscillates with the Alfvén frequency $\nu_A k_z$, since $\omega_m \rightarrow \nu_A k_z$, at large $|k_x(-t_0)|$. The other component of the magnetic mode $\psi_m^{(+)}$ as well as IMWs and SDWs are not excited initially, that is, their corresponding eigenfunctions are set to zero in the beginning, $\psi_m^{(+)}(-t_0) = \psi_{im}^{(\pm)}(-t_0) = \psi_s^{(\pm)}(-t_0) = 0$. These conditions ensure that initially only the magnetic mode is present, with a single direction of propagation, whereas IMWs and SDWs are absent. The corresponding initial state vector $\mathbf{h}_m$ associated with this magnetic mode is found as $\mathbf{h}_m(-t_0) = \mathbf{C}(-t_0) \cdot [0, \psi_m^{(-)}(-t_0), 0, 0, 0, 0]^T$, which is then inserted in equation (35) as an initial condition. For numerical integration we use a standard Runge-Kutta scheme (MATLAB ode45 RK implementation).

Figure 3 shows the subsequent evolution of the perturbed density, velocity and magnetic field components, while Fig. 4 shows the evolution of the corresponding mode eigenfunctions. In the beginning, being in the adiabatic region $|k_x(t)| \gg k_y$ and far from the coupling region, the initially imposed magnetic mode oscillates with the frequency ω_m , which at this large k_x is close to the Alfvén frequency, $\omega_m \approx k_z \sqrt{2/\beta}$, and induces similar oscillations in the velocity and magnetic field. The primary components in this mode are the azimuthal and vertical velocities, u_y and u_z , and magnetic field b_y and b_z , which oscillate with a nearly constant amplitude, whereas the density ρ as well as the radial velocity u_x and field b_x remain nearly zero, since the magnetic mode itself is incompressible.

The dominance of the magnetic mode and the absence of IMWs and SDWs during the early phase of the evolution are also evident

from the eigenfunctions in Fig. 4. Since the coupling matrix is small (Fig. 2), $\psi_s^{(\pm)}$, $\psi_{im}^{(\pm)}$ and the counter-propagating branch of the magnetic mode $\psi_m^{(-)}$ remain all very small compared to $\psi_m^{(+)}$, which oscillates with a constant amplitude according to equation (44).

As time progresses, $k_x(t)$ of the magnetic mode approaches and begins to cross the region $|k_x(t)/k_y| \lesssim 1$. In this region, the flow shear comes into play giving rise to two main effects: (i) the magnetic mode no longer oscillates but undergoes nonmodal MRI growth and (ii) the characteristic time-scales of the SDWs, IMWs, the MRI growth time and the shear time $1/q$ (in the non-dimensional form) all come close to each other, i.e., $\omega_s \sim \omega_{im} \sim \gamma_m \sim q$. Since the time-scales of the modes are comparable now, the coupling matrix elements become appreciable (Fig. 2). As a consequence, an efficient energy exchange – nonmodal couplings – among these modes as well as between the magnetic mode and the disc flow via MRI occurs during this stage. Note also that, in this interval, the adiabatic condition for the mode frequencies breaks down, $|d\omega(t)/dt| \sim \omega^2(t)$, which is another reason for mode coupling (Chagelishvili et al. 1996; Gogoberidze et al. 2004)³. Let us look in more detail at the evolution of variables and mode interactions in this coupling range.

First, at $t < 0$, when $k_x(t)$ enters the MRI-active range, $|k_x|/k_y \leq [(q\beta/k_z^2 - 1)(1 + k_z^2/k_y^2)]^{1/2} = 27.4$, or $|t| \leq 18.3$ (equation 31), the magnetic mode becomes non-oscillatory and starts to grow due to MRI, as seen in the evolution of $\psi_m^{(-)}$ in Fig. 4. The coupling term M_{12} becomes appreciable in this interval, resulting in the generation of the other component, $\psi_m^{(+)}$, of the

³ Note that this is not the case for small $k_y \ll k_z$, when the dynamics is uniformly adiabatic at all times even at $|k_x(t)/k_y| \lesssim 1$, as discussed in Section 2.2

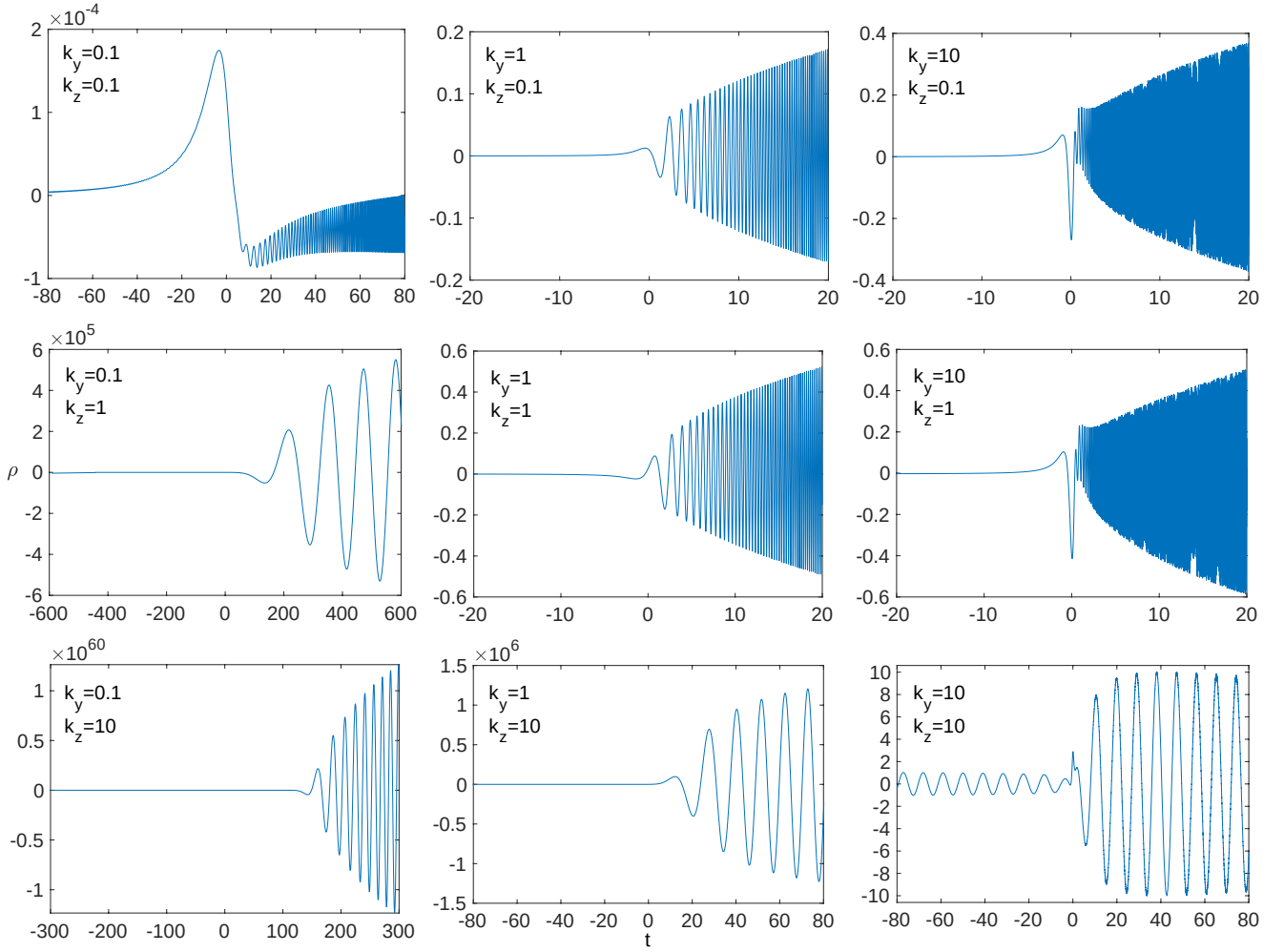


Figure 5. Evolution of density at different k_y and k_z for the same initial conditions consisting of a single magnetic mode component $\psi_m^{(-)}$. The SDW generation, indicated by high-frequency oscillations with increasing frequency and amplitude, tends to be more efficient at higher k_y and lower k_z , i.e., $k_z \lesssim k_y$.

magnetic mode by the initially imposed one $\psi_m^{(-)}$, as also shown in Fig. 4. The newly generated $\psi_m^{(+)}$ subsequently feeds back on the dynamics of $\psi_m^{(-)}$ through the coupling term M_{21} , which is also appreciable near $k_x(t) = 0$, thereby causing $\psi_m^{(-)}$ to decrease around $t = 0$. Afterwards both $\psi_m^{(-)}$ and $\psi_m^{(+)}$ grow due to MRI and mutual coupling, respectively, until they leave this MRI-active region. MRI in turn leads to the growth of all the velocity and magnetic field components. Since the coupling coefficients are larger now, the growth of $\psi_m^{(-)}$ also induces a moderate amplification of $\psi_s^{(\pm)}$ and $\psi_{im}^{(\pm)}$ due to the corresponding coupling terms. Then, at around $t = 0$, when $k_x(t)$ crosses the point $k_x = 0$, becoming positive $k_x(t) > 0$, qualitatively different behaviour emerges – high-frequency oscillations abruptly appear in the evolution of $\psi_s^{(\pm)}$ and low-frequency oscillations in $\psi_{im}^{(\pm)}$, indicating the generation of rapidly varying SDWs and slowly varying IMWs, respectively (Fig. 4). These very rapid oscillations are most evident in the evolution of ρ and u_x but are also seen in other components u_y , b_y , b_x and b_z , although the latter primarily oscillate with a lower frequency due to the magnetic mode and newly generated IMWs (Fig. 3). The energy needed for the SDW and IMW excitation is mainly extracted from the shear flow by the magnetic mode and transferred to SDWs and IMWs.

Then, $k_x(t)$ increases and moves in the next adiabatic region

$k_x(t)/k_y \gg 1$. The magnetic mode leaves the MRI-active region and the linear dynamics of SDWs, IMWs and the magnetic mode as well as the counter-propagating components for each of these modes become decoupled again, since all the elements of $\mathbf{M}$ are small again, tending to zero with increasing $|k_x(t)|$. As a result, there is no further energy exchange among the modes themselves and between the modes and the flow. The time-scales of the modes get separated again: $\psi_s^{(\pm)}$ and the associated ρ and u_x rapidly oscillate (Figs. 3 and 4) because of the newly excited SDWs with the frequency ω_s linearly increasing with time, while IMWs and the magnetic mode eigenfunctions, $\psi_{im}^{(\pm)}$ and $\psi_m^{(\pm)}$, continue to oscillate with the respective ω_{im} and ω_m frequencies (Fig. 4), which remain constant in time, $\omega_m \approx \omega_{im} \approx v_A k_z$. The other velocity and magnetic field components, being mainly related to the magnetic mode and IMWs, are accordingly dominated by these low-frequency oscillations, although still contain high-frequency contribution arising from SDWs whose amplitude are, however, much smaller than that of the low-frequency parts (Fig. 3), except in b_z where the contribution of rapid oscillations is larger (see equation 23). Note that in the incompressible case, at large $k_x \rightarrow \infty$, the longitudinal component u_x associated with the compressible SDWs are zero, since these waves are absent in that case, while the perpendicular components u_y, u_z and b_y, b_z are nonzero and

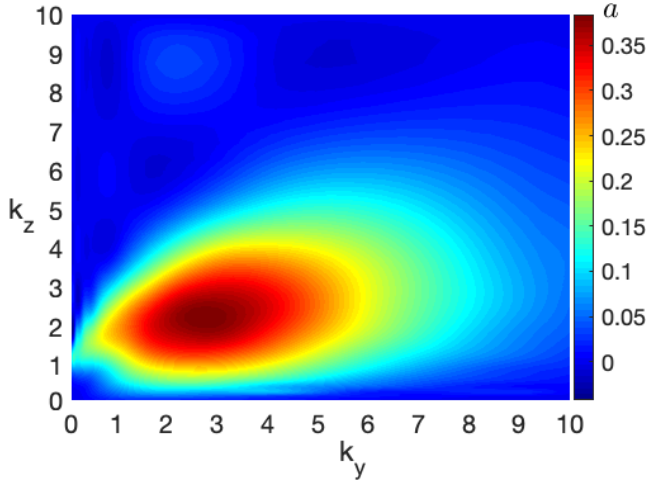


Figure 6. The amplitude a of generated SDWs for the density as a function of k_y and k_z . It achieves the maximum $a_m = 0.38$ and, hence the wave generation is most efficient, at $(k_{ym}, k_{zm}) = (2.8, 2.2)$ and decreases at smaller and larger k_y and k_z .

are associated with the incompressible magnetic and IMWs. This is the reason why the high-frequency oscillations of SDWs when they emerge in the present compressible case are most noticeable in ρ and u_x . Since our main goal is to study the generation of compressive SDWs by the incompressible MRI-unstable magnetic mode, below we further characterize this process as a function of azimuthal and vertical wavenumbers.

Finally, we point out that the generation of SDWs by the magnetic mode is mathematically a consequence of the well-known Stokes phenomenon, whereby different asymptotic WKB solutions of a differential equation emerge (i.e., change from being sub-dominant to dominant) as the complex (temporal) variable crosses anti-Stokes lines in the complex plane (Heading 1962; Berry & Mount 1972, see also HP09a for the case of vortex-SDW coupling). Physically, as mentioned above, it is caused by the nonmodal mechanism due to shear, which, changing k_x with time, gives rise to non-adiabatic regions with respect to the real variable (which is t or $k_x = qk_y t$ here), where time-scales of the modes and shear time become comparable to each other. Here we briefly illustrate this for SDWs, which are of main interest in this paper.

Let us consider the governing equations (42) and (43) for the eigenfunctions $\psi_s^{(\pm)}$ of SDWs from the full set of coupled modal equations (36). For simplicity, retaining only the dominant eigenfunction $\psi_m^{(-)}$ of the initially imposed magnetic mode on the right hand side of these equations, we get

$$\frac{d\psi_s^{(+)}}{dt} - (i\omega_s + M_{55})\psi_s^{(+)} = M_{52}\psi_m^{(-)}, \quad (45)$$

$$\frac{d\psi_s^{(-)}}{dt} + (i\omega_s - M_{66})\psi_s^{(-)} = M_{62}\psi_m^{(-)}. \quad (46)$$

In the initial adiabatic region at $t = -\infty$, there are no SDWs and the solutions of equations (45) and (46) are given by their particular solutions proportional to $\psi_m^{(-)}$, which accordingly oscillate with ω_m ,

$$\psi_s^{(+)} = -\frac{M_{52}}{i(\omega_s + \omega_m) + M_{55}}\psi_m^{(-)}, \quad (47)$$

$$\psi_s^{(-)} = \frac{M_{62}}{i(\omega_s - \omega_m) - M_{66}}\psi_m^{(-)} \quad (48)$$

and represent induced oscillations in $\psi_s^{(\pm)}$ by the magnetic mode. Since M_{52} and M_{62} decay at $|t| \rightarrow \infty$ (Fig. 2), both these solutions also decay with time. As k_x crosses the non-adiabatic region around $t = 0$, WKB regime breaks down, so this asymptotic solution is no longer valid and SDWs are excited due to the Stokes phenomenon. As a result, in the next adiabatic region at $t = \infty$, the solution is equal to the superposition of the excited SDWs oscillating with the high frequency ω_s , which represent homogeneous solution, and the former low-frequency particular solution originating from the magnetic mode (other contributions from IMWs are omitted here for simplicity),

$$\psi_s^{(+)} = a_s^{(+)} e^{\int_0^t [i\omega_s(t') + M_{55}(t')] dt'} - \frac{M_{52}}{i(\omega_s + \omega_m) + M_{55}}\psi_m^{(-)}, \quad (49)$$

$$\psi_s^{(-)} = a_s^{(-)} e^{\int_0^t [-i\omega_s(t') + M_{66}(t')] dt'} + \frac{M_{62}}{i(\omega_s - \omega_m) - M_{66}}\psi_m^{(-)}, \quad (50)$$

where $a_s^{(+)}$ and $a_s^{(-)}$ are the corresponding constant amplitudes of the excited counter-propagating SDWs. It follows from the form of the coupling matrix $\mathbf{M}$ (equation 37) that at $|t| \rightarrow \infty$, the real parts $\text{Re}(M_{55}) = \text{Re}(M_{66}) = -4.5\omega_s^{-1}(d\omega_s/dt) = -4.5d \ln\omega_s/dt$ and hence the SDW parts in solutions (49) and (50), after time integration of M_{55} and M_{66} in the exponent, scale as $\omega_s^{-4.5}$, which differs from the standard WKB-type dependence $\omega_s^{-1/2}$ (HP09a). This rapid decrease of the SDW parts at large times in $\psi_s^{(\pm)}$ is also seen in Fig. 4. In the standard situation, the WKB approach is applied to a second order differential equation and asymptotic solutions at $t = -\infty$ and $t = \infty$ are matched to find the amplitudes of generated waves. In particular, for the vortex-wave coupling in the simplest 2D case ($k_z = 0$), HP09a analytically found the amplitude of generated SDWs by the aperiodic vortical mode by matching WKB solutions across the anti-Stokes lines in the complex variable plane. However, in the present case analytically finding these amplitudes is complicated because of the complex form of a full set of modal equations (36) involving a larger number of interacting modes – the magnetic mode, IMWs and SDWs each with two counter-propagating components. For this reason, we adopt a different approach and quantify the SDW amplitudes – and hence their generation efficiency – using the density perturbation, in which, as noted above, these waves are most clearly manifested, rather than the asymptotic forms (49) and (50) of the SDW eigenfunctions.

3.1 Generation of SDWs by MRI for various k_y and k_z

Figure 5 shows the evolution of the density for the initially imposed magnetic mode as in Fig. 3, but for different k_y and k_z . For large $k_z = 10$ (i.e., when the vertical wavelengths of the modes are much smaller than the disc scale height), there are no rapid oscillations, implying that SDW generation does not occur even at higher $k_y = 10$; there are only low-frequency oscillations induced by the original magnetic mode and excited IMWs. The absence of SDWs in this case of large $k_z \gg k_y$ is because of the fact that the time-scales of SDWs and the magnetic mode remain well separated during an entire course of the evolution and the WKB regime is uniformly

valid at all times. A similar behavior is also observed at $k_y = 0.1$ and $k_z = 1$. By contrast, an efficient SDW generation takes place at smaller $k_z = 0.1, 1$ and larger $k_y = 1, 10$, or more generally $k_z \lesssim k_y$, where the favourable conditions for the mode coupling arise – existence of the non-adiabatic region at $|k_x(t)| \lesssim k_y$ where the time-scales of the SDW, IMWs, magnetic mode and shear time $1/q$ are comparable. This leads to an efficient generation of SDWs by the magnetic mode evidenced by the presence of high-frequency oscillation with increasing amplitude in ρ . A moderate SDW generation is also observed at small $k_y = k_z = 0.1$.

We can quantify the efficiency of the SDW generation by calculating wave amplitudes as a function of k_y and k_z . Taking into account the asymptotic expressions for SDW parts in $\psi_s^{(\pm)}$, which scale as $\omega_s^{-4.5}$, from equations (49) and (50) and the form of the transformation matrix $\mathbf{C}$ (equation B1) at large times $t \rightarrow \infty$, we obtain the asymptotic form of the high-frequency part of the density associated with SDWs

$$\rho = \omega_s^{1/2}(t) \left(a_1 e^{i \int_0^t \omega_s(t') dt'} + a_2 e^{-i \int_0^t \omega_s(t') dt'} \right), \quad (51)$$

whose magnitude thus increases with time as $\omega_s^{1/2}(t)$, as seen in Fig. 3. Here a_1 and a_2 are the constant in time complex amplitudes of each counter-propagating SDW branch near the generation point $t = 0$. Figure 6 shows the dependence of the total amplitude $a = (|a_1|^2 + |a_2|^2)^{1/2}$ of the waves, on k_y and k_z . It is seen that when these wavenumbers are large, $|a|$ is small and therefore the SDW generation is inefficient consistent with the behaviour in Fig. 5. With decreasing k_y and k_z , the SDW amplitude increases and attains its maximum value $a_m = 0.38$ at $(k_{y,m}, k_{z,m}) = (2.8, 2.2)$, indicating that the most efficient generation of non-axisymmetric SDWs by the magnetic mode takes place at azimuthal and vertical wavelength, $\lambda_{y,m} = 2\pi/k_{y,m} = 2.24$ and $\lambda_{z,m} = 2\pi/k_{z,m} = 2.9$, of perturbations comparable to the disc scale height H . This is consistent with the results of HP09a who showed that the SDW excitation by the vortical mode in the 2D case ($k_z = 0$) is most efficient at comparable, but somewhat smaller $k_{y,m} = 0.78$ (in units of H^{-1}), however, in the present case of SDW generation by the magnetic mode, it appears to be more efficient at higher $k_z \gtrsim 1$. The values of $k_{y,m}, k_{z,m}$ lie in the region $1 \lesssim k_y, k_z \lesssim 3$ where non-axisymmetric magnetic mode undergoes appreciable nonmodal MRI growth, as shown in Gogichaishvili et al. (2018, see their Figs. 1 and 2). This indicates that SDWs are excited most efficiently at wavenumbers for which MRI is still significant.

4 SUMMARY AND DISCUSSION

In this paper, we have revealed and investigated a linear process of generation of spiral density waves by MRI-unstable non-axisymmetric magnetic perturbations in compressible Keplerian discs with a nonzero net vertical background magnetic field in the shearing box approach. This wave excitation process is a consequence of a generic linear mode coupling phenomenon associated with the nonmodal dynamics of perturbations arising from the non-self-adjoint nature of shear flows, including Keplerian discs.

As is usually done in the shearing box, we have decomposed perturbations into spatial Fourier harmonics – shearing plane waves with a time-dependent radial wavenumber k_x – and analyzed their evolution and dynamics by numerically solving the linearized MHD equations for different azimuthal, k_y , and vertical, k_z , wavenumbers. We have derived the general dispersion relation in the WKB, or adiabatic regime, $k_y \ll k_z$, in order to classify

perturbation modes in the considered flow system. These modes are: SDWs driven mainly by compressibility and two incompressible modes – IMWs driven by rotation and magnetic tension force and the magnetic mode driven by magnetic tension. The latter two modes become standard Alfvén waves at large k_x . The magnetic mode plays a key dynamical role, since it is the mode that exhibits nonmodal MRI growth at smaller k_x . Using the canonical formulation, we have rewritten the main linearized MHD equations for the eigenfunctions of these modes first in the WKB regime when the equations diagonalize and the mode eigenfunctions decouple. Then, we have generalized these modal equations to the more important non-adiabatic regime, when they become coupled for non-axisymmetric ($k_y \neq 0$) eigenfunctions due to shear, giving rise to dynamical couplings among these three modes. Then, we have focused on the coupling of non-axisymmetric SDWs and the magnetic mode. To this end, we initially imposed the magnetic mode of a tightly leading orientation ($k_x/k_y < 0$, $|k_x| \gg k_y, k_z$) in the adiabatic regime and traced the subsequent evolution of density, velocity and magnetic field perturbations as well as mode eigenfunctions till the magnetic mode eventually becomes tightly trailing ($k_x/k_y > 0$, $k_x \gg k_y$) in the next adiabatic region. The magnetic mode, evolving in time, enters the non-adiabatic coupling interval ($|k_x| \lesssim k_y$), where it undergoes transient, or nonmodal MRI growth and, at crossing the point $k_x = 0$, abruptly generates SDWs and IMWs. The generated SDWs are evidenced by the appearance of high-frequency oscillations notably in the density and radial velocity perturbations, while the other components of velocity and magnetic field, being related to the magnetic mode and IMWs, oscillate with lower, of the order of the Alfvén frequency. The mode coupling in the non-adiabatic region occurs because the coupling terms between different mode branches become important due to shear and the characteristic time-scales of these modes come close to each other, enabling an efficient mode interaction in this interval.

We have also characterized the SDW generation as a function of k_y and k_z and found that it is most efficient when these wavenumbers $k_y, k_z \sim H^{-1}$, that is, when the azimuthal and vertical wavelengths of perturbations are comparable to the disc scale height. At these wavenumbers nonmodal MRI growth for non-axisymmetric magnetic mode is still appreciable, implying that the regimes of MRI and SDW generation overlap.

Because the strongest MRI–SDW coupling occurs at wavelengths comparable to the disc scale height, vertical stratification may therefore play a dynamical role. For the sake of mathematical simplicity, we have ignored vertical gravity and the resulting stratification of density and pressure, assuming that they are uniform in the vertical z -direction. This simplification allowed us to isolate and focus on the essential mechanism of the shear-induced generation of SDWs by the MRI-unstable magnetic mode. As shown in Mamatsashvili et al. (2013), stratification primarily modifies IMW mode, transforming it into inertia-gravity wave, while only weakly affects the magnetic mode, leaving its growth rate due to MRI essentially unchanged. Thus, although stratification may quantitatively modify the mode frequencies and eigenfunctions, it is not expected to alter the fundamental dynamics of the SDW generation process studied here. This process, being a consequence of the shear-induced linear mode coupling phenomenon, remains robust as it originates from the non-self-adjoint nature of the disc’s differential rotation. The SDW generation occurs mainly in the horizontal (k_x, k_y) -plane, where the effect of shear is concentrated, during a short time interval when $k_x(t) \approx 0$, although it still depends on k_z . By contrast, the effect of stratification is concentrated mainly in z . This timescale of the wave generation process is

much shorter than the buoyancy oscillation period associated with stratification.

The present analysis can be generalized to a vertically stratified case within the same shearing box approach, although it would be mathematically more involved. Specifically, in the presence of isothermal stratification, which yields a Gaussian density profile in z and is often adopted in MRI-turbulence studies (e.g., Davis et al. 2010; Guan & Gammie 2011; Bodo et al. 2014; Ryan et al. 2017; Held et al. 2024), a linear analysis of MRI can be conducted by expanding the variables in z on the basis of Whittaker cardinal functions instead of Fourier modes (Latter et al. 2010), while in the horizontal x - and y -directions the same decomposition in shearing waves can be applied. The order of the Whittaker functions, which is an integer number, determines the vertical scale of the perturbations and therefore plays a role analogous to that of the vertical wavenumber k_z in the unstratified case. It replaces k_z in the governing linear equations and the corresponding dispersion relation, as shown in Latter et al. (2010). Thus, the main modification introduced by vertical stratification is that shearing waves of perturbation modes are characterized by discrete vertical mode numbers, while retaining a time-dependent $k_x(t)$ and constant k_y . Therefore, these stratified shearing waves are expected to exhibit qualitatively similar dynamics – nonmodal MRI growth and mode coupling – to those in the unstratified case studied in this paper. In particular, the dependence of the amplitude of the generated SDWs on k_y and vertical mode number in the stratified case will be qualitatively similar to that in Fig. 6, but, of course, with some quantitative differences.

Finally, a connection of the present results with studies of MRI-turbulence in compressible discs is in order. SDWs have often been observed in simulations of MRI-turbulence with a nonzero net magnetic flux, where they typically appear as large-scale structures, comparable to the box size, in the density distribution and also contribute to Reynolds stress (Fromang & Stone 2009; Nelson & Gressel 2010; Blaes et al. 2011; Flock et al. 2011; Gressel et al. 2012; Bai & Stone 2013; Zhu et al. 2013; Ross & Latter 2018; Flock et al. 2017; Sun & Bai 2021). The production of SDWs and associated density fluctuations by MRI-turbulence have been attributed in those studies to the linear vortex-wave coupling mechanism of HP09a. As noted above, this mechanism is essentially hydrodynamic in nature, relies on the conservation of potential vorticity, and is two-dimensional, i.e. uniform in the vertical direction, as was also confirmed by the same authors in their subsequent nonlinear simulations of zero net flux MRI-turbulence (Heinemann & Papaloizou 2009b). However, in the 3D MHD case with nonzero net flux, the situation is fundamentally different in that potential vorticity is no longer conserved and the MRI-unstable magnetic mode dominates the dynamics, complicating the identification of the pure vortical mode in the turbulent state. As a result, SDWs excited by MRI-turbulence are not always uniform, but rather exhibit some large-scale variation in the vertical direction. The generation of SDWs by the magnetic mode demonstrated in this paper can be regarded as another main process that can produce compressible motions via MRI-driven mostly incompressible velocity perturbations in nonzero net flux MRI-turbulence. For this reason, it is expected that SDW amplitudes and resulting density fluctuations will be stronger than those produced by vortices, although to verify this further nonlinear study of compressible nonzero net flux MRI is required. This may be particularly important for disc heating, since SDWs generated in MRI-turbulence can steepen into shocks, thereby dissipating energy (Rafikov 2016; Kaul et al. 2025). The stronger the wave amplitude

at the generation, the more intensive the heating is due to shock dissipation. In addition, the azimuthal and vertical wavelength of the highest wave generation efficiency identified in this study will be important in guiding future nonzero net flux MRI-turbulence simulations to optimally capture SDWs generation process.

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6 DATA AVAILABILITY

The data underlying this article will be shared on a reasonable request to the corresponding author.

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APPENDIX A: EXPRESSION FOR THE EVOLUTION MATRIX $\mathbf{A}$

Substituting the variables $(h_1, h_2, h_3, h_4, h_5, h_6)$ into equations (21)-(23), yields the following explicit form of the evolution matrix $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k_z^2 & 0 & k_x k_z & 0 & k_y k_z & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_x k_z & 0 & 2q - \frac{2k_z^2}{\beta} - \left(1 + \frac{2}{\beta}\right) k_x^2 & 0 & -\left(1 + \frac{2}{\beta}\right) k_x k_y & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ k_y k_z & 0 & -\left(1 + \frac{2}{\beta}\right) k_x k_y & -2 & -\frac{2k_z^2}{\beta} - \left(1 + \frac{2}{\beta}\right) k_y^2 & 0 \end{pmatrix}, \quad (\text{A1})$$

which enters equations (32) and (35).

APPENDIX B: EXPRESSION FOR THE TRANSFORMATION MATRIX $\mathbf{C}$

The eigenvectors of the evolution matrix $\mathbf{A}$ are obtained by substituting its eigenvalues $\pm i\omega_m, \pm i\omega_{im}, \pm i\omega_s$, as given by dispersion relation (27), in equation (34) and, for a chosen normalization, have the form $[f_1(\omega), i\omega f_1(\omega), f_2(\omega), i\omega f_2(\omega), f_3(\omega), i\omega f_3(\omega)]^T$, where $\omega = \pm\omega_m, \pm\omega_{im}, \pm\omega_s$ and the functions f_1, f_2 and f_3 are

$$f_1(\omega) = \frac{k_z[k_x f_2(\omega) + k_y f_3(\omega)]}{k_z^2 - \omega^2},$$

$$f_2(\omega) = \omega^4 - (k_y^2 + k_z^2) \left[\omega^2 \left(1 + \frac{2}{\beta} \right) - \frac{2k_z^2}{\beta} \right],$$

$$f_3(\omega) = 2i\omega(\omega^2 - k_z^2) + k_x k_y \frac{\omega^4 - f_2(\omega)}{k_y^2 + k_z^2}.$$

The columns of the transformation matrix $\mathbf{C}$, as defined in the Section 2.2, are constructed from the eigenvectors of $\mathbf{A}$ and can be written in the following order

$$\mathbf{C} = \begin{pmatrix} f_1(\omega_m) & f_1(-\omega_m) & f_1(i\omega_{im}) & f_1(-i\omega_{im}) & f_1(\omega_s) & f_1(-\omega_s) \\ i\omega_m f_1(\omega_m) & -i\omega_m f_1(-\omega_m) & i\omega_{im} f_1(i\omega_{im}) & -i\omega_{im} f_1(-i\omega_{im}) & i\omega_s f_1(\omega_s) & -i\omega_s f_1(-\omega_s) \\ f_2(\omega_m) & f_2(-\omega_m) & f_2(i\omega_{im}) & f_2(-i\omega_{im}) & f_2(\omega_s) & f_2(-\omega_s) \\ i\omega_m f_2(\omega_m) & -i\omega_m f_2(-\omega_m) & i\omega_{im} f_2(i\omega_{im}) & -i\omega_{im} f_2(-i\omega_{im}) & i\omega_s f_2(\omega_s) & -i\omega_s f_2(-\omega_s) \\ f_3(\omega_m) & f_3(-\omega_m) & f_3(i\omega_{im}) & f_3(-i\omega_{im}) & f_3(\omega_s) & f_3(-\omega_s) \\ i\omega_m f_3(\omega_m) & -i\omega_m f_3(-\omega_m) & i\omega_{im} f_3(i\omega_{im}) & -i\omega_{im} f_3(-i\omega_{im}) & i\omega_s f_3(\omega_s) & -i\omega_s f_3(-\omega_s) \end{pmatrix} \quad (\text{B1})$$

It can be verified that with this form of $\mathbf{C}$, the spectral decomposition $\mathbf{A} = \mathbf{C}\mathbf{\Lambda}\mathbf{C}^{-1}$, holds, which is used in the main analysis. As before, $\mathbf{\Lambda}$ is a diagonal matrix composed of the eigenvalues of $\mathbf{A}$, i.e., $i\omega_m, -i\omega_m, i\omega_{im}, -i\omega_{im}, i\omega_s, -i\omega_s$, which are ordered corresponding to the columns of $\mathbf{C}$,

$$\mathbf{\Lambda} = \begin{pmatrix} i\omega_m & 0 & 0 & 0 & 0 & 0 \\ 0 & -i\omega_m & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega_{im} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\omega_{im} & 0 & 0 \\ 0 & 0 & 0 & 0 & i\omega_s & 0 \\ 0 & 0 & 0 & 0 & 0 & -i\omega_s \end{pmatrix} \quad (\text{B2})$$